Next Generation Models for Convertible Bonds with Credit Risk

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1 Introduction

Convertible bonds are hybrid securities which offer equity-like returns when the share of the issuing firm is strong, yet behave like conservative fixed-income investments when the stock market is either stagnant or negative. Indeed the convertible bond is essentially a bond that can be converted into shares, a feature which allows the equilibrium of interests between the three parties involved, the issuing company, the equity investor and the fixed-income investor to be struck more efficiently than was the case when equity and fixed-income were treated as separate investment categories, involving different, if not incompatible, standards. As the company issuing the convertible bond sells an embedded option to convert into its shares, it expects its creditors to charge a lower fee than would otherwise apply to its credit class, hence is able to pay lower coupons. Reciprocally, the fixed-income investor earning these coupons is rewarded by his upside participation in the performance of the share. The equity investor, on the other hand, whose basis of judgment is the price of the equity and its expected return, makes up for the premium paid over parity by the downside protection that the bond floor automatically provides, and by the fact that the convertible bond coupons are usually set higher than the projected dividends (hence creating the notion of a break-even date).

It is obvious, from this preliminary analysis, that the equity level will be the determining factor in the convertible bond value, where "value" means what specifically distinguishes the convertible bond from an ordinary fixed-income investment or an ordinary equity investment. This is the reason why the quantitative analysis of convertible bonds lends itself naturally to the Black-Scholes analysis where the share price is the state variable, and dynamic hedging strategies are the basis for the valuation of the embedded option. Not only will the convertible bond value depend on the volatility of the share, but we shall expect the share price itself to set the dividing line between equity behavior and bond behavior (as well as between the ensuing concerns, respectively about share price volatility and credit quality volatility). It turns out indeed that the Black-Scholes analysis provides just the right unifying framework to formulate the convertible bond pricing problem. Unification comes at a cost, however. For we can no longer ignore, once the share becomes the driving factor, what direct effect it may have on the issuer's credit quality. And if such an effect is to be assumed, we can no longer but model it explicitly.

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2 Credit spread and the fixed-income logic

The quantitative measure of credit quality has traditionally been the credit spread. Risky bonds are priced by the market at a discount to sovereign debt, and the price difference, when expressed in terms of the excess in the implied yield, is the credit spread. Bonds maturing at different dates can imply different credit spreads, hence creating credit spread term structure. When the bonds are zero coupons, the term structure of credit spread is equivalent to giving the whole array of risky discount factors, etc. So credit spread is really a notion from fixed-income analysis, and for that reason quite foreign to a framework such as Black-Scholes, where the state variable is the underlying share. As long as bond pricing was the sole concern, all that the fixed-income analyst needed was the spot yield curve and the spot credit spread curve. The necessity of modeling stochastic credit spread, however, became evident with the emergence of credit derivatives. But when the derivative payoff did not specifically depend on the credit spread (for instance options on corporate bonds), a risky yield curve model could still be developed along the lines of the traditional yield curve models (Hull and White, Black Derman Toy, Heath Jarrow Morton), in such a way that the changes of credit spread curve and the changes of risk-free yield curve would indistinguishably be captured by the overall changes of the risky yield curve. Only when credit spread changes had to be separately modeled did the need arise to identify the real "physical" variable underlying these changes, the instantaneous probability of default of the issuer (assuming a deterministic recovery rate). Just as the instantaneous interest rate is the state variable driving the basic yield curve models (e.g. Hull and White), the instantaneous probability of default, or hazard rate, drives the stochastic credit spread models. One writes directly the stochastic process followed by the hazard rate (possibly with a time dependent drift in order to match a given spot credit spread curve) and generates different prices for risky zero coupon bonds in different states of the world, in other words, different credit spread curves.

3 Credit spread and the convertible bond

The relation between the convertible bond and the credit spread seemed at first to arise only from the "bond character" of the convertible. The embedded equity option would be priced in the Black-Scholes framework alright, where discounting takes place at the risk-free interest rate, but the presence of a fixed-income part of course implied that "something" had to be discounted under a risky curve, if only to be consistent with the fixed-income analysis of the issuer's debt. The difficulty, however, was that bond component and equity component were not readily separable. On the contrary, we saw that the convertible bond could very well display a mixed behavior, now like equity, now like bond, depending on the share level. This is why the question "How exactly to apply the credit spread in the convertible bond pricing tree, and how to link that to share price?" became the central problem in convertible bond valuation.

One early attempt interpreted the mixed behavior of the convertible bond in probabilistic terms (Goldman, 1994). It is only with *some probability*, so the argument went, that the convertible bond would end up like equity or end up like pure bond, and that probability was identified with the probability of conversion. Ignoring what distortions might arise from other embedded options, such as the issuer's call or the holder's put, the suggestion was to discount the value of the convertible bond, at nodes of the pricing tree, with a weighted average of the forward instantaneous risk-free interest rate and the forward instantaneous risky rate. The delta of the convertible bond, now identified with the probability of conversion, would determine this weighting. While this approach certainly fulfilled the wish that the convertible bond be treated as equity when most likely to behave like equity, and as bond when most likely to behave like bond, it certainly did not explain the financial-theoretic meaning of the mixed discounting. Perhaps it can be argued, in a global CAPM framework, that some future cash flow ought to be discounted with some exotic mixture of some given discount rates. The problem is, no sense can be made of a situation where the mixing takes place *locally*, and the weighting varies from one state of the world to the other.

More recently, another approach (Tsiveriotis & Fernandes 1998) thought better to interpret the mixed behavior of the convertible in *actuality* rather than *potentiality*. If the convertible bond is really a combination of a bond and an equity option, why not actually treat it like one, and split it into two components, one to be discounted risk-free and the other risky? When there is involved the possibility of early call or early put, however, the two discounting procedures cannot take place completely separately, so what T&F have proposed is to throw into the bond component whatever value accrues from the issuer's liability (either promised or contingent cash flows), and into the equity component whatever value accrues from the holder's contingent claim to convert into the issuer's share or from the issuer's *own* contingent claim (the idea being that the issuer could always deliver his shares, default or no default, and that he would not exercise his option to call back the convertible in case of default or shortage of cash). The two components are priced as two distinct assets. The equity component has the equity or nothing payoff as termination value, and the bond component (or the cash-only component, as T&F labeled it) the cash or nothing payoff. The two backward recursions are then coupled through the following algorithm. In case the convertible bond checks for early conversion—or early call– in a certain state of the world, the equity component is set equal to conversion value—or early redemption value—in that state, while the cash-only component is set to zero; alternatively if the convertible bond checks for early put, the cash-only component is set equal to the put strike and the equity component is set equal to zero. The cash-only component, on the other hand, earns the coupons in all states of the world.

It is interesting to note the circularity, or self-reference, that is inherent in both approaches. The value of the convertible bond crucially depends on the proportion in which the credit spread is applied to it, yet this proportion ultimately depends on the convertible bond itself. In the first approach the proportion is determined by the delta which is itself a derivative of the convertible bond value, and in the second, the proportion is determined by the value of the cash-only component relative to the equity component, which in turn depends on the particular constraint that the convertible bond as a whole checks, the conversion constraint, the call constraint or the put constraint. Mathematically, this translates into non-linearity. This is the reflection of the fact that the risky component of the convertible, which is the value that the holder is liable to lose in case of default—and which, by the same token, he will argue he is entitled to recover a fraction of when the assets of the defaulted company are liquidated —, depends on the optimal behavior of the holder himself[1]. While the recovery entitlement of the holder of a straight bond is a straightforward fraction of the present value of the bond, the holder of an option-embedded bond, such as the convertible bond, will typically want to recover more, for he will invoke what contingent rights he was holding on top of the fixed ones. And this notably depends on his optimal exercise or conversion policy in case of no default.

4 The missing story of default

However, this whole explanation in terms of loss and recovery in case of default is totally missing from the T&F paper. As a matter of fact, the problem with the T&F approach is that it falls one step short of telling the whole story about the convertible bond under default risk. While it certainly proposes an *actual* splitting of the convertible bond into two distinct components, and reproduces its desired extreme behaviors (pure bond, pure equity) at the extremes of the share price range, it does not say what actually *happens* to the convertible bond in case of default. And default can take place anywhere between those two extremes. T&F's line of argument is simply to identify the cash-only component (why not through a complex procedure involving two pricing PDEs and their local coupling), then to uncontroversially apply to it the credit spread, in the old fixed-income logic.

A few paragraphs back, we argued that if one wishes to model the actual default process and not just describe its phenomenological consequence, the credit spread, one has to get hold of the real physical variable underlying it, the hazard rate. All the more so when a pricing framework, such as Black-Scholes, already imposes on us a reduction in terms of state variables. Now it certainly makes sense to choose a credit spread of some given finite maturity, say one year, as state variable, and develop a stochastic model for credit quality in the same vein as the so-called "market models" of interest rates. Furthermore, one can assume some explicit correlation between the share process and the credit spread process and complete the program that we have announced earlier, of explicitly modeling the effect of the issuer's share on his credit quality. The T&F approach would generalize to this framework, the tree of the cash-only component would become two-dimensional and discounting would take place under local credit spread. However, this would still not answer the question why the cash-only component has to be discounted with credit spread in the first place, any better than just the postulation that it somehow condenses the issuer's liability and that credit spread should mimetically apply to it.

To our eyes, a model that relies on surface resemblance and no real explanatory argument is not a satisfactory model. Lying at the crux of the T&F model is the proposition that the equity component and the cash-only component are two identifiable, if hypothetical, contingent claims, hence should follow the Black-Scholes PDE, the one with risk-free discounting and the other with risky discounting. Now the Black-Scholes PDE is not just a pricing black box. It relies on "physical" first principles which are the continuous hedge and the no-arbitrage argument. The causal explanation of the Black-Scholes PDE is the precise elaboration of *that which happens* to the hedge portfolio over the infinitesimal time increment dt. On the other hand, given that the whole idea of the T&F splitting is to model the split behavior of the convertible bond under default risk, then why not go all the way, and try to spell out exactly what can happen to the convertible bond in the eventuality of default? For the convertible bond is the real contingent claim after all. T&F's progress relative to the Goldman paper was that they took the step from *probabilistically* mixing two distinct credit stories that are likely to happen to the same instrument, to *actually* decomposing that instrument into two distinct credit entities. So what we are now urging is that the credit story really be told about those two credit entities.

5 The story of default told at last

"What can happen in case of default" is a question hinging *directly* on the probability of default. It imposes the instantaneous probability of default, rather than *effects* thereof such as the credit spread, as the original cause — or true explanatory variable. The extension to the case of the convertible bond under credit risk, not of the Black-Scholes PDE, but of *the line of reasoning* underlying the Black-Scholes PDE, would then run as follows.

Calling p the instantaneous probability of default (or the intensity of the Poisson default process), what "infinitesimally" happens to the composite portfolio, convertible bond and dynamic hedge, under default risk is:

a) with probability 1−pdt: no default. The usual Black-Scholes continuous hedge argument applies, the holdings in the underlying share are chosen is such way that the hedge portfolio is immune to market risk over the time increment dt, and infinitesimal P&L can be written as (assuming no dividends for simplicity):

$$\delta\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

b) with probability pdt: default. The infinitesimal P&L is literally swamped by the loss of the defaultable fraction X:

$$\delta\Pi = -X$$

The expected P&L is then expressed as follows, neglecting second order terms:

$$E(\delta\Pi) = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - pX\right) dt$$

If we now assume that the probability of default is given in the risk neutral world[2], we can equate the above expectation with the risk-free growth of the portfolio:

$$E(\delta\Pi) = r\Pi dt$$

and obtain the following PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV + pX \qquad (1)$$

6 Differences explained

Once the problem of convertible bond valuation under default risk is framed in such a unifying formalism, the differences between all the models that the practitioners have been using with more or less rigor find an explanation in terms of different choices of X.

6.1 Grow risky, Discount risky

Let X be the whole portfolio, or in other words let us assume that both the convertible bond and the underlying share drop to zero in case of default, and the PDE will transform into:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r+p)S\frac{\partial V}{\partial S} = (r+p)V$$

This corresponds to the popular model, mnemonically known as “Grow risky, Discount risky.”

6.2 The general model

A more general model is one in which the share drops to a residual value $(1-\eta)S$ upon default, and the convertible bond holder is entitled to recovering a fraction F of his investment. He would then have the option either to convert into shares at their residual value, or to recover F. In this case, X would be expressed as:

$$X = V - \max[\kappa(1-\eta)S, F] - \frac{\partial V}{\partial S}\eta S$$

where κ is the conversion ratio, and the PDE governing the convertible bond value under default risk would become:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r+p\eta)S\frac{\partial V}{\partial S} = (r+p)V - p\max[\kappa(1-\eta)S, F] \qquad (2)$$

The question remains how to model F. Should it be a fraction of the face value N of the convertible bond? Or should it be a fraction of the market value of the corresponding straight bond: what the practitioners call its “investment value”? To be exact, the recovered fraction should be established by the liquidator after default has taken place. However, we can assume an *a priori* recovery rate applying uniformly to the issuer’s liabilities, whatever their nature, certain or contingent. What you recover is proportional to what you are owed. The holder of a coupon-bearing bond is owed more than the holder of a zero coupon bond, hence should recover more. And the holder of a bond with an embedded option, say a put, is owed more than the holder of a bullet bond, hence should recover more. The concept of Probability, according to Ian Hacking, emerged from those gambling situations where the players were for some reason prevented from pursuing the game until the end. The game had to be settled one way or the other, and the money at stake distributed according to some rationale. This is how the notion of a player’s best chances of winning it first made its appearance, or in other words, his expected gain. Settling the case of default of a convertible bond issue is no different. What the holder is supposed to recover in case of early termination due to default is the recovery fraction of the expected value of the cash flows he would otherwise get in case of no default.

7 Modeling the cash claim of the convertible bond holder

So what is it exactly that the holder of the convertible bond is owed prior to default?

7.1 The N-model

If you say it is a fraction of the face value N, then the PDE would look something like that:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r+p\eta)S\frac{\partial V}{\partial S} = rV + p(V - \max[\kappa(1-\eta)S, RN])$$

Let us call it the N-model.

7.2 The Z-model

If you say it is a fraction of the present value of the outstanding coupons and face value, then you would have to determine first whether this present value should be computed under risky or risk-free yield curve. It all depends on the interpretation of that which “the convertible bond holder is owed prior to default.” Since we are in the business of building a mathematically consistent model of the fair value of the convertible bond and we believe, for that matter, that the market is the fairest dispenser of value, then a possible interpretation of “the value that the holder of a convertible bond is owed prior to default” could simply be the fair value, or market value, of the convertible bond itself! And this market value would already have the default risk factored in it. Therefore a somewhat extreme model would be one where the holder simply recovers a fraction of the convertible bond value prior to default (cf. the paper by A. Takahashi et al. 2001):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r + p\eta) S \frac{\partial V}{\partial S} = rV + p(V - RV)$$

The reason why this is not satisfactory is that we had assumed on the other hand that the holder would still have the right to convert into the residual value of the underlying share upon default, so the PDE would really have to look like:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r + p\eta) S \frac{\partial V}{\partial S} = rV + p\,(V - \max[\kappa(1-\eta)S, RV])$$

The recovery rate of the underlying share $(1-\eta)$ and the recovery rate of the convertible bond R being completely independent, we would then be faced with the possibility that $\kappa(1-\eta)S$ may be greater than RV, even though $\kappa S \le V$ at all times. Nothing would then guarantee that the holder may not optimally elect to convert into the residual value of the share, over and above the fact that the value he is recovering anyhow, the recovery fraction of the *convertible* bond, already incorporates the value of a conversion right! While the recovery procedure is aimed at compensating the convertible bond investor in case of default, we certainly do not suppose that it ends up doubling his conversion rights! Conversely, if RV is much greater than $\kappa(1-\eta)S$, say $R = 1$ and $\eta = 1$ in an extreme case, it would also seem strange that the holder should recover the full convertible bond value (including the full value of the conversion right) when the share has actually dropped to zero! To sum up, if we are keen on leaving the holder the right to convert right at the time of default, then the only way to avoid this conflict is to assume that the value he is likely to recover, or in other words that which "he was owed prior to default," has been stripped of the conversion rights first.

So it seems we are back to modeling F as the present value of the underlying straight bond, and the above digression would have only convinced us that the fair value of "what the holder is owed prior to default" should take into account default risk, or in other words, that the present value of the outstanding coupons and face value should be computed under the risky curve. Thus we have:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r + p\eta) S \frac{\partial V}{\partial S} = rV + p(V - \max[\kappa(1-\eta)S, RZ])$$

where Z, the present value of the straight bond, solves the same PDE as the convertible bond, only without conversion rights:

$$\frac{\partial Z}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Z}{\partial S^2} + (r + p\eta) S \frac{\partial Z}{\partial S} = rZ + p(1-R)Z$$

If the hazard rate were independent of S, the last PDE would integrate to:

$$Z(t) = \sum_{t_i \ge t}^{T} C_i e^{-\int_t^{t_i} (r + p(1-R))\,du}$$

where C_i are the outstanding cash-flows. (This is simply a forward calculation under the risky yield curve). Otherwise we would have to solve in parallel two full PDEs, Let us call this model the Z-model.

7.3 The *P*-model

So far we have considered two interpretations of the notion of recovery. These corresponded to two different interpretations of the notion of default of the convertible bond. In one case, the convertible bond was construed as a debt instrument, binding the issuer to redeem the principal at maturity and to pay interest in the meantime. Default in this case meant that the *structure* of the convertible bond as debt instrument was over, and that the investor had to be reimbursed the amount of money he had initially invested. Recovery would then simply appear as a case of early redemption, caused by default, and it would mean recovering a fraction of the principal right away. In the other case, the convertible bond was construed as a tradable asset whose fair market value represented all the value there is to consider, and default simply meant that this value had dropped to zero. Recovery would then amount to recovering part of the pre-default holdings, or in other words, a certain fixed proportion of this value. So in the one case, default is a failure of a contractual obligation while in the other, it is simply a failure of market value. Recovery is defined accordingly: in one case the holder is owed the principal while in the other he is owed this market value, and the difference between the two interpretations is further reflected in the fact that the recovered value in the first case is purely nominal and independent of market conditions, while in the other, it is itself subject to interest rates and credit spread discounting. The two models further differ in that the N-model does not really discriminate between the holder of a zero-coupon bond and the holder of a coupon-bearing bond as far as recovery is concerned, while the Z-model does. However, the two models have in common that the holder is offered an amount of cash (RN or RZ) right after default and right before he exercises his last option to convert, and that that is the end of the story.

Now consider a refinement of the N-model where we wish to compensate the holder of a coupon-bearing bond more than the holder of a zero-coupon bond. What should he recover exactly? We cannot just pay back a fraction of the sum of the face value and the outstanding coupons because the coupons were just the reflection of the scheduling of the issuer's debt over *time*. A more appropriate model seems to be one where the holder recovers a fraction of the *present value* of the outstanding straight bond, where this present value is *discounted under the risk-free curve*. In a sense, the occurrence of default eliminates default risk, and we wake up the day after in a default-free world where this present value calculation is the only way to discriminate between the holder of a zero coupon bond, and the holder of a coupon-bearing bond. Hence the following PDE:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r+p\eta)S\frac{\partial V}{\partial S}=rV+p\left(V-\max[\kappa(1-\eta)S, RP]\right)$$

where:

$$P(t)=\sum_{t_i\geq t}^{T} C_i e^{-\int_t^{t_i} r(u)du}$$

Let us call this the P-model.

Although it looks as if the P-model is just intermediate between the N-model and the Z-model (it achieves more than the N-model in integrating the coupons but achieves less than the Z-model in not applying full discounting with the credit spread), in fact it opens a whole a new perspective for it is the first among the models we've considered so far to assume that life continues after default, and to bring the post-default world into the picture. Indeed the discounting of the recovered value under risk-free curve, simple as it may seem, is in fact an instance of a general category of models which we will examine later and which *couple the pre-default world and the post-default world*.

8 The optimal model

For now let us just note that a common point between all the previous model models is that none of them involved non-linearity such as alluded to earlier. The situation is somewhat akin to a free-boundary problem. In all the cases above, we imposed on the convertible bond PDE that the recovery fraction be some value computed separately. Be it a fraction of the face value, or of the present value of the outstanding payments, we never let F be determined freely by the value of the convertible bond itself. As mentioned previously, the holder of a risky bond with an embedded option will want to argue that he was owed more prior to default than just the present value of the fixed income part of the bond. Having agreed to exclude the option to convert from the treatment of recovery, this means that contingent cash-flows such as puts and calls have ideally to be incorporated in the holder's claim to recovery. The problem is that their precise value will depend on whatever optimal exercise policy the holder was supposed to follow prior to default. Just as the free-boundary problem inherent in American option pricing translates into maximizing the value of the American option and the early exercise boundary is itself part of the solution (see Wilmott 1998), we feel that the fraction of the convertible bond with other embedded options that the holder will ideally want to claim for recovery, is the greatest such fraction subject to the constraint that it may be legally argued, once default has taken place, that this fraction was owed to the holder. We are implying, in other words, that our algorithm for computing the recovery fraction F should really act as a lawyer trying to optimize his client's interests, and that the real lawyers should perhaps equip themselves with our convertible bond pricing model under default risk, once it is completed, in order to best serve their client. And just as the free-boundary problem is essentially non linear, we should expect ours to be non linear.

8.1 Our proposed model: the AFV splitting

Trying to bring together all the desiderata and the constraints that our "philosophical" analysis of default and recovery seems so far to suggest for the case of the convertible bond, we can summarize them as follows:

- Split the convertible bond value into two components: $V=B+C$.
- B is the value that the holder will argue he was owed anyway prior to default, and consequently will claim he must recover a fraction of according to some universal recovery rate R. Hence $F=RB$, in our case.
- B will be worth at least the present value of the underlying straight bond, for the holder will typically argue that he was owed more than this present value in case of an embedded options such as a put.
- B should not include the option to convert. On the contrary, the option to convert acts "externally" to the process of recovery, for the holder will retain the right to convert at the residual value of the share once default *and* recovery have taken place.
- C would then have to incorporate this option to convert, and would consequently finish as the holder's last option to convert into the residual value of the share when default takes place.

Given our general PDE for the convertible bond:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r+p\eta)S\frac{\partial V}{\partial S}=(r+p)V-p\max[\kappa(1-\eta)S, RB]$$

subject to the constraints of early call and early put:

$$V\geq \max(B_p, \kappa S)$$
$$V\leq \max(B_c, \kappa S)$$

(where B_p is the holder's put strike price, and B_c the issuer's call price, $B_c>B_p$), the following coupled PDEs should in effect be solved in order to value the convertible bond under default risk:

$$\frac{\partial B}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 B}{\partial S^2}+(r+p\eta)S\frac{\partial B}{\partial S}=(r+p)B-pRB$$

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+(r+p\eta)S\frac{\partial C}{\partial S}=(r+p)C-p\max[\kappa(1-\eta)S-RB, 0]$$

with initial conditions:

$$B(S,T)=N$$
$$C(S,T)=\max(\kappa S-N, 0)$$

and subject to the following algorithm (which is the cause of non-linearity):

- If $B_p > \kappa S$ and the continuation value of $B + C$ is less than B_p then $B := B_p - C$
- Else if $B_p \leq \kappa S$ and the continuation value of $B + C$ is less than κS then $C := \kappa S - B$
- If $B_c < \kappa S$ then $C := \kappa S - B$
- Else if $B_c \geq \kappa S$ and $B + C$ is greater than B_c then $C := B_c - B$
- $B := B +$ Coupon, on coupon dates

Notice that the term that multiplies the hazard rate in the right hand side of each PDE expresses the recovery value of each one of the two components after default. For the bond component B, this is the usual term, whereas for the option to convert, or equity component C, this is the intrinsinc value of the holder's last option to convert into the residual value of the share.

9 Interpretation of the T&F model in our framework

We argued earlier that T&F do not provide a justification of their mathematical model in terms of what happens *in effect* to the convertible bond and to its components in case of default. Their splitting is just a heuristic splitting which tries to fulfill at best the desiderata that we have listed above, to the effect that the bond component should capture the cash-flows, fixed and contingent, that the holder is owed, and the equity component should capture his right to convert, etc., only it stops short of telling *the whole, consistent story of default*. What we call "telling the whole, consistent story of default" is that we be able to write PDEs for B and C that govern their respective values prior to default *by way of explicitly stating the outcome of default for these values*. So it certainly would be interesting to try to test T&F's model against our criterion.

The general PDE that the convertible bond value solves in the T&F model is the following:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV + pB$$

It falls in our general schema (1) with $X = B$, and it splits into:

$$\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} + rS\frac{\partial B}{\partial S} = (r + p)B$$
$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} = rC$$

with initial conditions:

$$B(S, T) = \begin{vmatrix} N & \text{if } \kappa S \leq N \\ 0 & \text{otherwise} \end{vmatrix}$$
$$C(S, T) = \begin{vmatrix} 0 & \text{if } \kappa S \leq N \\ \kappa S & \text{otherwise} \end{vmatrix}$$

and subject to the following algorithm:

- If $B_p > \kappa S$ and the continuation value of $B + C$ is less than B_p then $B := B_p$ and $C := 0$
- Else if $B_p \leq \kappa S$ and the continuation value of $B + C$ is less than then $B := 0$ and $C := \kappa S$
- If $B_c < \kappa S$ then $B := 0$ and $C := \kappa S$
- Else if $B_c \geq \kappa S$ and $B + C$ is greater than B_c then $B := 0$ and $C := B_c$
- $B := B +$ Coupon, on coupon dates

Notice that T&F do not assume that the underlying share drops in the event of default, and that they assume zero recovery.

If we were to recount the consequences of a default event on a convertible bond holder, as this transpires through the T&F model, we would have to admit that he first loses B, *and second*, that he carries on holding the Black-Scholes asset C which is unaffected by default. In other words, life continues after default in the T&F model through the subsequent trading and hedging of the equity component C. See in comparison how life stops in the AFV model in case of default: the holder has to make a last optimal decision, either to exercise the right to convert at residual value or to recover a cash amount. *And notice that both the bond and equity component are subject to default risk in the AFV model:* they both undergo a jump, the bond component to its recovery value, and the equity component to its intrinsic value.

10 The coupling of pre-default and post-default worlds

Now it would certainly make sense to imagine a continuation of life after default. A softer appellation of the state of default would be "distress regime," and a more general model would be one where the holder may have to reserve until later his decision to convert at the post-default value of the share. Indeed it may not be optimal to exercise the option either to recover the cash value or to convert at residual value of the share, right after default. Cases were witnessed where the conversion ratio was revised after default. Not to mention that the volatility of the underlying share is most likely to have dramatically changed too. Therefore, a more accurate description would be one where the holder ends up holding an "ersatz-convertible bond" in case of default, with bond floor equal to RB, underlying share spot level equal to residual value, possibly a different conversion ratio (provided the issuer agrees to postpone the reimbursement of the recovered value), all of which would have to be priced in a new world (i.e. a new PDE), where volatility may be different, and, last but not least, where credit risk is different. Indeed an open question is whether the post-default world is not default-free. Could a company that has already defaulted default once again on the recovery value of its previous debt? And if it does, wouldn't that mean that the post-default world itself has to be further coupled with a post-default-post-default world?

Assuming for simplicity that default happens only once, or in other words that the distress regime is default-free, the general convertible bond pricing model we are contemplating may now be expressed in the following terms:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r+p\eta)S\frac{\partial V}{\partial S}=rV+p[V-V'(S(1-\eta),t)]$$

with the usual convertibility, puttability, and callability constraints:

$$V\geq \max(B_p,\kappa S)$$
$$V\leq \max(B_c,\kappa S)$$

and initial condition:

$$V(S,T)=\max(\kappa S,N)$$

$(V-V')$ is the jump that the convertible bond value undergoes in the event of default, and the jump into default is now generally seen as a case of switching to the distress regime.

10.1 Case of no life after default

If life ends with the default event, then $V(S,t)$ has to assume one of the following "stopped" solutions:

- $V'(S_\tau,\tau)=\max(\kappa S_\tau,RN)$: N-model
- $V'(S_\tau,\tau)=\max\lfloor\kappa S_\tau,RZ(S_{\tau^-})\rfloor$: Z-model
- $V'(S_\tau,\tau)=\max\lfloor\kappa S_\tau,RB(S_{\tau^-})\rfloor$: AFV model

where τ is the time of default and $S_\tau=S_{\tau^-}(1-\eta)$.

10.2 Case of life after default

Otherwise, if V' is allowed to have a life after default, we may write for it the following, default-free, PDE:

$$\frac{\partial V'}{\partial t}+\frac{1}{2}\sigma'^2 S^2\frac{\partial^2 V'}{\partial S^2}+rS\frac{\partial V'}{\partial S}=rV' \quad (3)$$

Note that the volatility of the share in the distress regime has possibly new value σ'.

Imposing the right constraints and the right initial and boundary conditions on this PDE, will depend on the policy that the issuer wishes to pursue after default.

10.2.1 Suppose he agrees to pay the remaining fraction of coupons and face value at their pre-default payment dates, but grants a conversion option just at the moment of default and not after, then the ersatz-convertible bond V' will solve PDE (3) with

- the following initial condition:

$$V'(S,T)=RN$$

- the following jump-conditions on coupon dates:

$$V(S,t^-)=V(S,t^+)+R\,Coupon$$

- and the following "time of default" constraint:

$$V(S,\tau)\geq\kappa S_\tau$$

where τ is the time of default.

In other words, we would just have the P-model.

10.2.2 Suppose the issuer extends the conversion option and that he maintains the original scheduling of the interest payments (to be applied now to the recovered fraction). The ersatz-convertible bond V' will solve PDE (3) with, in this case,

- the following initial condition:

$$V(S,T)=\max(\kappa S,RN)$$

- the following jump-conditions on coupon dates:

$$V'(S,t^-)=V'(S,t^+)+R\,Coupon$$

- and the following continuous constraint:

$$V(S,t)\geq\kappa S$$

So really the ersatz-convertible bond will behave like a mini-convertible bond in this case, with a new bond floor and initial underlying value equal to the recovery value of the share $S(1-\eta)$. (We are of course ignoring how the embedded put or call options would fare under the new distress regime). This model really looks like the P-model, only the option either to convert into the recovery value of the share or to recover a fraction of the outstanding straight bond has been given time value.

10.2.3 Suppose the issuer extends the conversion option but doesn't want to postpone the payment of the cash recovery fraction B. V' will now solve PDE (3) with

- the following initial condition:

$$V'(S,T)=\kappa S$$

- the following continuous constraint:

$$V'(S,t)\geq\kappa S$$

- and the following "time of default" constraint:

$$V'(S,\tau)\geq RB(S_{\tau^-},\tau^-)$$

where τ is the time of default, and B the risky component. This constraint expresses the fact that the holder has to make the optimal decision, at the time of default, whether to accept the recovery cash value RB and end the game, or to go on holding his option to convert in the life after default. However, due to the martingale property of the underlying asset, the solution of PDE (3) with boundary conditions such as described, collapses to:

$$V'(S(1-\eta), \tau) = \max[RB(S, \tau^-), \kappa S(1-\eta)]$$

So really this case would be equivalent to the AFV model.

10.2.4 Finally, suppose that the issuer extends the conversion option after default, only it is an option with a very strange terminal payoff and knock-out barrier.

V' solves PDE (3)

- with initial condition

$$V'(S, T) = \begin{vmatrix} 0 & \textit{if } \kappa S \leq N \\ \kappa S & \textit{otherwise} \end{vmatrix}$$

- and boundary condition:

$V'(S, t) = 0$ for all S and t such that it is optimal for the CB holder to exercise the put

$V'(S, t) = \kappa S$ for all S and t such that it is optimal for the CB holder to convert the bond

As for the fraction B that is lost on default and likely to be partially recovered, it solves the following risky PDE

$$\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} + (r + p\eta)S\frac{\partial B}{\partial S} = (r+p)B$$

Only suppose that it is subject to the following, no less puzzling, initial and boundary conditions:

$$B(S, T) = \begin{vmatrix} N & \textit{if } \kappa S \leq N \\ 0 & \textit{otherwise} \end{vmatrix}$$

and $B(S, t) = 0$ for all S and t such that it is optimal either for the issuer to call the bond, or for the holder to convert it.

Bringing the pieces together, the convertible bond value would be governed by the following PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r + p\eta)S\frac{\partial V}{\partial S} = rV + p[V - \max(RB, V'(S(1-\eta), t)]$$

and it would switch to the following PDE in case of default and in case $V'(S(1-\eta), \tau) > RB(S, \tau)$ at the time of default:

$$\frac{\partial V'}{\partial t} + \frac{1}{2}\sigma'^2 S^2 \frac{\partial^2 V'}{\partial S^2} + rS\frac{\partial V'}{\partial S} = rV'$$

where V' is this strange knock-out equity option.

11 Conclusion: A philosophical refutation of T&F?

When $R = 0$ and $\eta = 0$ the mathematics of the last model (10.2.4) becomes identical to T&F. They both give the same value for the convertible bond and its components. All we have done really is interpret the formalism of T&F in our general philosophy where the actual consequences of default are spelled out exactly. An interpretation cannot prove a theory right or wrong. It only gives us arguments to accept it, or to prefer another theory to it. The T&F model is mathematically consistent. It produces the kind of behavior the trader expects from the convertible bond at the extremities of the stock price range. However our general presentation of the various recovery models has convinced us by now that there is much leeway in the choice of model for B, the cash claim of the holder in the event of default.

Much as it seemed legitimate that the cash claim, or the value recovered, in the AFV model should depend on the optimal behavior of the holder in case of no default (remember the case for the cash settlement of interrupted gambling games), we see no reason why it should so dramatically depend on such a hypothetical behavior in the T&F case, as to deny him *any* cash recovery claim in those regions where he would have optimally converted. Even less so do we see the reason why the contingent claim that the holder ends up holding in the life after default should be knocked out in those regions where he would have optimally exercised the put.

T&F would of course object that we are over-interpreting their model. It is only when viewed in the perspective of the post-default world, that the Black-Scholes component C and the cash claim B look so strange! For if one were to stop at the surface, and envision the split into B and C as just a rule for varying the weight of the credit spread in the overall discounting procedure of the convertible bond value, then all that would matter is that the credit spread be applied in the "right places", and this is certainly what T&F achieves! The reason why we feel uncomfortable with this minimalist demand, however, is that we do not think we can possibly shy away from the consequences of the default event. B is the risky component in the T&F model; hence B is the fraction that I expect to lose in case of default. And if I don't lose everything then I keep something of some value, and then I have to explain why this something has this value. The only explanation is that I can cash in immediately this something through some action (either actual cash, or conversion on the spot), or that this something is just the present value of something that lives through future actions and decisions...

FOOTNOTES & REFERENCES

1. The delta-weighting approach is a somewhat heuristic, probabilistic expression of the same thing.

2. Alternatively, we can argue, with Wilmott (1998) and Merton (1976): that if the jump component [of the asset price or the convertible bond price] is uncorrelated with the market as a whole, then the risk in the discontinuity should not be priced into the option. Diversifiable risk should not be rewarded. In other words, we can take expectations of this expression and set that value equal to the risk-free return from the portfolio. This is not completely satisfactory, but is a common assumption whenever there is a risk that cannot be fully hedged; default risk is another example of this. (Wilmott 1998, p. 330). (Note: In this paragraph, Wilmott is in the process of deriving the option pricing PDE in the presence of jumps in the underlying; but like he says, the reasoning applies to default risk too).

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