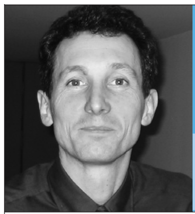


Nail in the Coffin

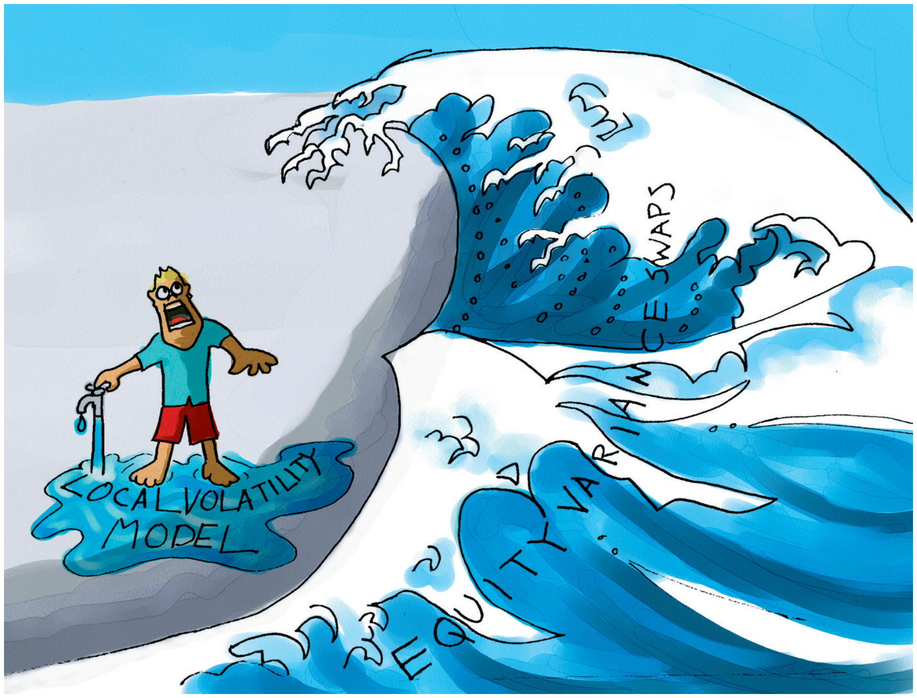
How Exotic Is the Variance Swap?

The local volatility model helped give birth to a liquid equity variance swap market. But, **Philippe Henrotte** writes, the ungrateful child has now matured and is about to terminate its creator ...*

A Popular Instrument

The variance swap has recently attracted considerable interest among the equity desks. The growing popularity of this instrument may be explained by its unique features which appeal to various people for different reasons.

Option traders appreciate the smoothness of the variance swap. Unlike calls and puts, it offers a volatility exposure which does not depend on the relative position of the spot with some arbitrary strike, and it does not need to be constantly delta hedged. The variance swap behaves a lot like a long maturity option, with its large and smooth Vega and its small Gamma. But long dated equity options are illiquid and they obviously become short dated at some point. Every option trader has experienced the way a benign vanilla option turns into a hellish time bomb as the spot hits the strike price close to maturity. Traders with limited supply of adrenaline like the way variance swaps die in a whisper.



Risk managers make an additional critical observation. They view the variance swap as a light exotic because, for a large class of models, theory suggests that it can be replicated by a portfolio of vanilla options. Most remarkably, the replication strategy is both static and model independent. With option prices readily available, pricing and hedging a variance swap is therefore no longer an issue and risk managers have given *carte blanche* to their trading desks to write large quantities of swaps. Unlike calls and puts, the variance swap has the advantage of not spreading its liquidity along the strike dimension. Today it is seriously competing with the vanilla options to become the benchmark derivative security on the equity market.

The equity quants like the variance swap because it vindicates their favorite local volatility model, a technology which has been prevalent among equity desks for the last ten years. Indeed, only jumps would invalidate the static replication of the variance swap with vanilla options, and the local

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volatility model is happily free of jumps. Since the local volatility model is able to reproduce any option smile pattern, the variance swap falls nicely within its scope. Observed deviations between market quotes for variance swaps and the value of their static replications are explained by a lack of information on the illiquid deep out of the money puts. Reverting the argument as often in finance, variance swaps are seen as completing the option market by providing an indirect quote on the log contract which never took off as a stand-alone derivative.

In a previous "Nail in the Coffin" piece, my colleague Elie Ayache described the irony in the risk manager's point of view. The replication argument gave birth to a liquid instrument, but as its market matured, it freed itself from the matrix of the theory. As Elie puts it, "the liquidity of the instrument insinuates itself underneath the theoretical cover-up and reveals the true face of the instrument." You could call this the Frankenstein's syndrome. We show in this short note how this irony extends to the quant's point of view. A frontal collision is looming between the variance swap and the local volatility model. The variance swap is too strong today to gently step aside because it no longer fits the quant's agenda. As in a bad horror movie, the beast is about to devour its creator.

The jumps, stupid!

The most compelling case for the presence of jumps is probably the substantial short maturity skews of implied volatility smiles, both for indices and for single stocks. Only unrealistically large short term local volatilities can account for these skews while reasonable jumps produce them naturally.

Short term barriers critically depend on the jump assumptions and their market quotes would most likely rapidly invalidate the local volatility framework. The irony principle failed however for equity barriers because no convenient replication story could be told under some realistic set of assumptions. Equity traders were never able to manage a risk which they could not fully control and their barrier market remains today largely illiquid. With no reference one-touches in place to contradict them, local volatility models produce unrealistic barrier prices on which risk manager impose huge safety spreads. This sorry state of affair is a far cry from the sophistication of the foreign exchange market where liquid one-touches have served as reference benchmarks for many years.

In a influential research note on variance swaps, Demeterfi and al. (1999) show that in absence of jumps the fair variance delivery price K_{var} of a variance swap with maturity T is given by the formula

$$\begin{aligned}\frac{TK_{var}}{2} &= rT - \left(\frac{S_0}{S}e^{rT} - 1\right) - \log(S_0/S)\\ &+ e^{rT}\int_0^S \frac{1}{K^2}P(K)dK + e^{rT}\int_S^0 \frac{1}{K^2}C(K)dK,\end{aligned} \quad (1)$$

where S_0 is the current spot, S an arbitrary spot level, often conveniently chosen to be the forward price of the underlying for maturity T, and $P(K)$ and $C(K)$ are respectively the European call and put with strike K and maturity T. In practice, the two integrals are evaluated by some quadrature on a finite number of strikes. If the two integrals are well behaved for small and large strikes respectively, selecting a starting point for the puts and an end point for the calls should not be an issue. Practitioners report however that the lowest available strike in the market (we shall refer to it as K_L) is often too large to qualify as a decent numerical lower strike limit for the integral on the puts. This, they reason, explains why the theoretical fair price K_{var} deviates from its corresponding market quote. This numerical behavior is however consistent with large negative jumps that would allow the spot to fall below K_L with substantial probability. This would indeed render the out of the money put much more valuable than any diffusion model would suggest. The integral

$$\int_0^S \frac{1}{K^2}P(K)dK$$

in Equation 1 should then be numerically evaluated with very small strikes indeed. In the extreme case of default where the spot jumps to zero, it is easy to see that the put $P(K)$ should be at least as valuable as the discounted strike times the risk neutral probability of default. This yields an integral of $1/K$ which diverges to infinity in zero.

Demeterfi et al. (1999) do not deny that jumps invalidate the exact replication argument. If the squared log returns are used to compute the infinitesimal contributions to the overall variance, and assuming that the spot follows a simple jump diffusion with a jump of fixed relative size y and intensity λ, one can show that the theoretical fair price K_{var} of a variance swap deviates from the diffusive formula in Equation 1 by a correction term C given by

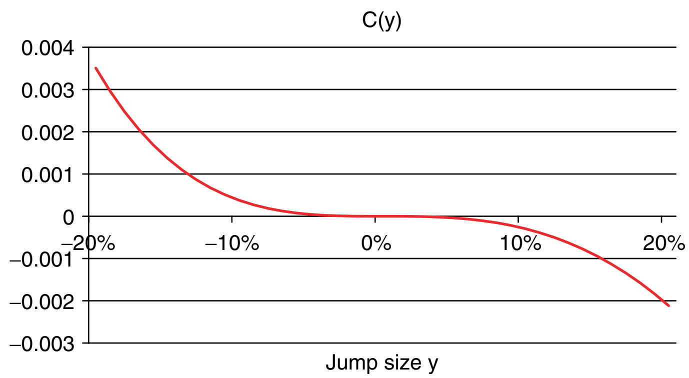
$$C = \lambda\left[(\log(1+y))^2 - 2\log(1+y) - 2y\right] - \tfrac{1}{3}\lambda y^3 + O(y^4) \quad (2)$$

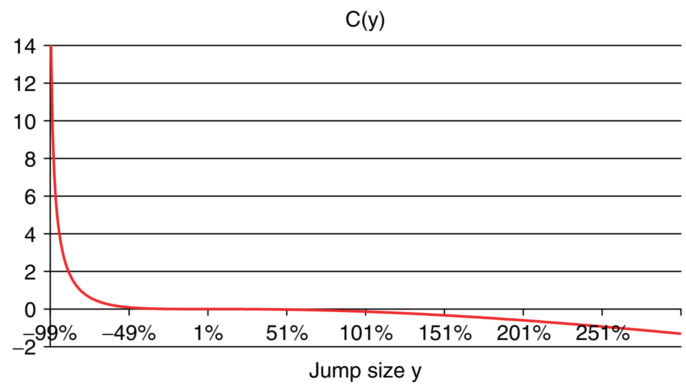
The diffusive formula underestimates the theoretical fair variance delivery price when the jump is y negative. The leading cubic term of $C(y)$ means that the correction term is not significant when jumps are small. For large jumps, it can be argued that negative and positive jumps would possibly cancel each other, resulting in an overall small effect. This last line of reasoning should be analyzed with care. Although the leading term in the error term is $C(y)$ indeed signed and symmetric, the subsequent terms in the expansion have alternating signs so that a large positive jump has a much smaller effect than a negative jump of the same amplitude. If we limit ourselves to jumps y between -20% and $+20\%$, the function $C(y)$ for say looks $\lambda = 1$ indeed fairly symmetric as can be seen in Figure 1.

If we now allow the jump size y to vary between -99% and $+300\%$, the same function $C(y)$ in Figure 2 does not look symmetric anymore.

A negative jump of -50% yields an error term $C(y) = 0.094$ while a positive jump of size $+50\%$ yields $C(y) = -0.025$, with a negative jump of -60% we have $C(y) = 0.207$ while a positive jump of size $+60\%$ yields only $C(y) = -0.039$.

Large jumps are therefore critical in the valuation of variance swaps. Two remarks are in order here. First it is not the frequency of the jumps which matters for the valuation of the variance swaps but the price that market participants are willing to pay to seek protection against these bad





states of nature. One should therefore not be surprised if the risk neutral jump intensity is a lot larger than its statistical counterpart.

Second serious models should attempt to capture the stochastic nature of the jumps since there is no reason to believe that the market will keep its jump prediction fixed. Whereas short term options only depend on the short term jump predictions, the long maturity variance swap will integrate through time the stochastic behavior of the jumps. Forward starting variance swap should help calibrate the process of the jump parameters.

Spot homogeneity

Between the trader, the risk manager and the quant, only the trader has the right intuition and a valid argument. The variance swap is indeed a convenient spot homogeneous instrument, which makes it an ideal candidate for the calibration of spot homogenous models. Inversely, the spot inhomogeneous calls and puts are the natural calibration instruments of the inhomogeneous local volatility model.

Whereas spot homogeneous models can deal with calls and puts and have been shown to fit market smile surfaces, local volatility model are tailored to uniquely fit the smile. The variance swap prices produced by the local volatility model are bound to collide with the market quotes which do integrate the possibility of jumps. Local volatility models will not survive this collision because, unlike in the market for barriers, they will not be able to hide in a convenient fog of large spreads. Variance swaps are already liquid and their spreads are tight. This is the quant's irony.

Having buried the local volatility concept, one cannot help but go one step further in the de-construction of the equity derivative market. We conjecture that the vanilla options bring little additional value once a vigorous variance swap market is in place. Variance swaps and their associated derivatives such as forward starting swaps and options on variance offer a full set of tools on which to calibrate a complex homogenous model with stochastic volatility and jumps. There may come a day when options will no longer be liquid but only custom tailored for specific client needs. It was probably unfortunate that derivative markets started thirty years ago with vanilla options and not with variance swaps. These inhomogeneous instruments have lead everyone in the impasse of inhomogeneous local volatility models. Who would seriously have thought of a local volatility model with variance swaps traded and no option on the horizon?

An exotic conclusion

Before we attempt to decide if the variance swap is exotic, we should first agree on the definition of an exotic instrument. We could say that an instrument is exotic if it cannot be replicated by a static portfolio of options. Following this traditional definition, we saw that we do not agree with the conventional wisdom which views the variance swap as a light exotic. Jumps are a fundamental ingredient in the valuation of the variance swap and they certainly cannot be hedged with a convenient static portfolio of options.

More fundamentally we see no reason to accept the paradigm which puts the options before any other derivative in a virtual pecking order for contingent claims. We expect that the market will decide soon who takes precedence among the derivatives. The dominance of the inhomogeneous species and their associated models may well be comparable to the reign of the dinosaurs. And a vanilla call may soon be considered exotic.

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