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Single-Case Statistics?

The myth of ex-ante valuation

It is the mortality tables that first put the idea in our minds that a single definite value, the life insurance premium, can be associated with a statistical population. The notion of *present value* is the main result of actuarial science. In this article, we argue that this statistical valuation does not require the notion of probability or, generally, an ex-ante stance.

The insurance contract that is written for an individual person is not "priced" in a forward-looking fashion, as it would have been in a credit market where the collapse, or "death," of a given entity is "priced." Even though it may look as if the insured person was not dead already and the *probability* of her future death was the only issue; even though it may look as if the insurance company was dealing with a single person and a single-case event, in reality, the probability is only the reflection of an observed, that is to say, of a *past and settled,* frequency. Better, it is a number implied from an ex-post accounting equation.

In reality (or rather, in our narrative), entire populations have died and for long periods of time the insurance company *has not* broken even. Generation after generation, it has slowly adjusted the insurance premium it would charge for the next cohort until its *ex-post returns* finally equaled zero. All it did is therefore to solve an accounting problem. The single-case probability is only the reformulation in terms of the average person of the break-even that the insurance company has managed to reach in its overall *ex-post* accounts. We should always look at the actuarial valuation problem ex-post, never ex-ante.

In its statistical interpretation, probability presupposes the whole population and, more importantly, the fact that different individuals have been recognized to belong to the same population to begin with (the so-called *reference class*). Accordingly, the probability of the single-case event is not defined.[1] Although it shows up as an integrand in the formalism, in reality, probability presupposes the integral. It is only a way of rephrasing a statistical problem in the singular or, equivalently, of disguising an ex-post adjustment as an ex-ante forecast. How? When the insurance premium that makes the insurance company break even on average is finally found, the "probability" of a single death is simply defined as the number to plug into the formula *in retrospect*, in order that the "expected life" and the corresponding capitalization of the known insurance premium yield the right mathematics.

The notion of present value thus has nothing *present* about it. It is only after observing different individuals die at different ages and generate a distribution of losses that the idea occurred to the insurance company to *represent* or synthesize this population, using the notion of the representative individual. Since a living person of a certain age could not simultaneously be all the persons who had been known to die at later ages, simply the idea was to make those persons *succeed to the representative one in time*, one at a time, so that the singular person may *later* turn out to be any single one of them. The only way to make an individual represent an entire population is to *create* the notion of probability.

What we are saying is that the notion of precedence in time (and thus of a *present* value), or the notion that the representative person would *later* die at a certain age with a certain probability, was imposed on us logically by the need to map the one into the many. As a matter of fact, the representative "person" could have remained forever the mere synthesis of an already-dead population; it could have remained the ex-post average of essentially dead people; it could have remained multiple. It is only when this synthesis was identified with *one* living person – a person all the more alive and kicking as it was claiming insurance – that the notion of probability was forced on us.

Probability is past, not future, and is only *misplaced* in the future. Correlatively, it turns the reality of ex-post accounting into the myth of ex-ante valuation. This suggests the rather unusual thought that if the "present value" of a certain contingent claim were available per se – somehow "naturally" attaching to the single case and not requiring a synthesis or a reduction from a whole statistical population – then probability wouldn't be needed and, more amazingly, we wouldn't even have to stage the whole ex-ante outlook.

The miracle of money

In truth, there is no such thing as *ex-ante* (literally: "before the event"). Real events are unforeseeable, not because of the indeterminism of the outcome within a list of identified and referenced possibilities, but because the real singular event belongs to no series and to no reference class. Real events create their own causes or the *possibilities that will have led to them*, as Bergson would say.[2] Nassim Taleb calls this the *backward narrative*.[3] It is only after the event that we can think of it and of what went before.

How to be able to predict the event or even look at it, when the event and its whole context (or reference class) are not available beforehand? The probability that a certain member of a given population dies makes sense, as we have said, only because that member belongs to that population and because the so-called "prediction" concerning her death is in reality only a retro-diction – the false presentation as "present valuation" of what is merely the ex-post solution of an accounting problem.

The metaphysics of possibility has made us wrongly believe in the existence of an ethereal medium connecting the present spot with the future event, a medium whose currency, or numeraire, is measured in "probability." In reality, there is *nothing* to mediate between the present spot and the future event – no exchange, no currency. There is only time, but this time is void and has no relation with the event. Scrutinizing the insurance case, we notice that it is only money that provided the link and made it look as if a certain insurance premium was *presently* invested in a future contingent claim. We claim that *tense*, or the illusion of a dramatization in time (expectation, present value, etc.), is in fact derivative on money, which is here the crucial concept.

Recall that the insurance premium is an ex-post concept. It reflects a break-even, a statistical accounting equation finally zeroed. Now, it so happens that the money that the insurance company had to pay eventually *was also present* in its accounts before the events of dying actually took place. It was present, not because the company had "priced" the event before the event, but because of the long succession of trials and errors that we have already mentioned and because of the capacity of tweaking successfully the ex-post returns, thanks to the persistence of the statistical regularity. As a consequence, an amount of money, which is in fact the ex-post outcome of a long history of adjustments, suddenly manifested the miraculous property of subsisting in the account, *ahead* of the next generation of deaths.

Money is material and it *counts*; probability doesn't. It does makes a difference, at the time when the contingent event occurs and the insured amount changes hands, what premium was initially charged for the contingent claim. By contrast, it makes no difference what probability was initially estimated for the event. This should give us a hint as to how to "account" for the event before it occurs.

Note that the account of the insurance company is the medium that enabled this back and forth movement between the past and the future. It is the accounting equation of the insurance company that allowed us to represent a statistical distribution by means of a single-case problem and consequently to speak of an ex-post result as the "present value" of a future contingency. The decisive observation is that this "valuation" of contingency was made possible *immanently*, by the persistence of the statistics and of the account, not by a transcendent a priori valuation principle involving objective probability or a data-generating process, etc. From there, the idea naturally emerges of valuing general contingent claims in their appropriate immanent medium. This medium of contingency, I hold, is the market.

The reality of the market

It is the second thesis of this article that the market is the only way of pricing a single-case event (or the contingent claim written on it). This is possible through the miracle of money, not the miracle of probability. The market retains the unit of count that was crucially operative in the insurance case,

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namely money – the market is *financial* – however, it replaces the statistical tables (which had granted the insurance company the mysterious capacity of finding, or "calculating," the premium immanently) with something else. It replaces them with the *capacity to exchange* the contingent claim. The exchange is the immanent valuation principle of absolute *contingency*, or contingency that is single-case and irreducible to a reference class. The exchange is our "single-case statistics."

Our bold thesis is that the market is the way of dispensing with the mediation of probability when dealing with single-case events – or events whose reference class is not identified.[4]

We should generalize the immanent principle that is inherent in statistics directly to the single-case contingency, not reutilize the transcendent principle of probability, which is derivative and too special anyway. Simply, instead of breaking even on the long run (thanks to the temporal loop made possible by the statistical regularity and the closed-circuit account of the insurance company), we should break even directly in an "open account." This is what it means to value the contingent claim at its market price and consequently to be able to exchange it or unwind it at that price. This should even act as the definition of the market; that is, the material sheet on which the contingent claim is written, the market where it is exchanged for a price, and money which provides the unit of count were precisely invented to serve that purpose.

Bruno de Finetti had also sensed the key financial element involved in linking present value and future contingency. However, he mistook the byproduct (probability) for the main problem and sought only the meaning of the single-case probability. No wonder his transcendent escalation led him to *subjective* probability or an interpretation of probability as the betting odds offered for transaction by a single agent.[5] He couldn't have considered the market of the corresponding contingent claim because he thought there was no meaning in lending a probability evaluation to the market. But the real question is: Why focus on probability at all?

The case for an *objective* single-case probability was independently argued by Popper in his *propensity theory*.[6] He believed the event to be "generated" by a *generating condition* and the latter to be endowed with the propensity, or tendency, to later produce the observed frequency in case the experiment was repeated. In our modern financial parlance, this is tantamount to assuming the existence of a *random generator*.

Unfortunately, the propensity theory founders against the *reference class problem*. Indeed, the probability of the single event will depend on the reference class to which it is assigned rather than on the event itself. As Gillies explains, "the probability of a particular man aged 40 living to be 41 [...] will vary depending on whether we regard the individual merely as a man or more particularly as an Englishman [and] will alter depending on whether [he] smokes two packets of cigarettes a day." As a result, the propensity theory was ultimately said "to introduce a subjective element into the singular probability," if only because of the "doubt about the way we should classify the event."[7]

Similarly, when valuing derivatives in probability theory, we first have to agree whether the event triggering the derivative payoff is a member of a class in which the underlying price is the only variable, or a larger class in which the volatility of the underlying also varies, or a larger class still in which the underlying not only diffuses with Brownian volatility, but also admits of discontinuous jumps whose frequency and magnitudes may also vary, etc. By contrast, when derivatives are recognized as absolute contingent claims and are turned over to the market, their price is given immanently and absolutely, without presupposing a specific class of events on which to define probability or a statistical population on which to compute totally foolish actuarial averages.

Why should it be so difficult to admit that the market is an objective evaluator of contingent claims, and therefore an absolute substitute both to objective probability and to statistics? Instead of worrying about fair value in the long run or about a subsistent random generator, why not simply worry about the next market price? This imposes as sole principles: 1) nonarbitrage; 2) the imperative that derivative, underlying, and money account shall all be priced in the same marketplace. The shocking consequence is that derivative pricing disconnects totally from the "real objective probability" (be it statistically or propensity based) and introduces *risk-neutral probability* instead.

In reality, we shouldn't even call this a probability, as it is only the re-expression of our two principles above. There will be as many risk-neutral probability measures as there are ways that the market prices the underlying, the derivative, and the riskless bond consistently. This is the consummation of the thought that the instant market is the only "long run."

ENDNOTES

1. Richard von Mises, the major advocate of the statistical or frequency interpretation of probability, writes: "We can say nothing about the probability of death of an individual, even if we know his condition of life and health in detail. The phrase 'probability of death,' when it refers to a single person has no meaning at all for us" (von Mises, R., *Probability, Statistics and Truth*, 2nd revised English edition, Allen and Urwin, 1957).
2. Bergson, H. *Key Writings*, Ansell-Pearson, K. and Mullarkey, J. (eds), Continuum, 2002.
3. Taleb, N. N. *The Black Swan: The Impact of the Highly Improbable*, Random House, 2007.
4. Ayache, E. *The Blank Swan: The End of Probability*, Wiley, 2010.
5. de Finetti, B. *Theory of Probability*, vols. 1 and 2, English translation, Wiley, 1974.
6. Popper, K. R. Realism and the Aim of Science, Routledge, 1983.
7. Gillies, D. *Philosophical Theories of Probability*, Routledge, 2000, pp. 119–120.